AP-Observation Automata for Abstraction-based Verification of Continuous-time Systems

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Our goal

- Given

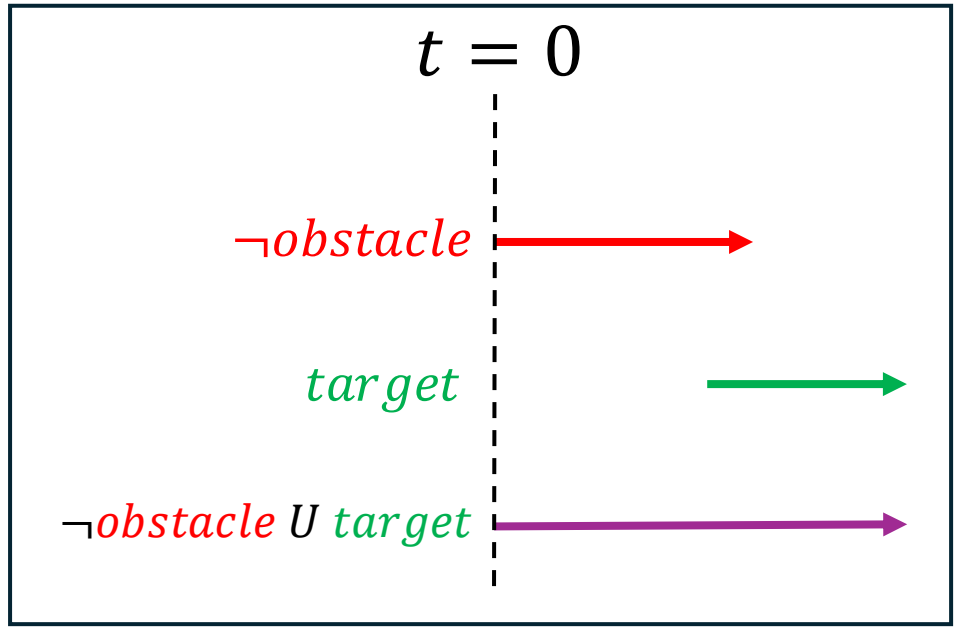
continuous-state
continuous-time
system Σ

LTL Specification
φ

- Problem: Does system Σ satisfy φ?
- Goal: An algorithm to verify if a given Σ satisfy a given φ

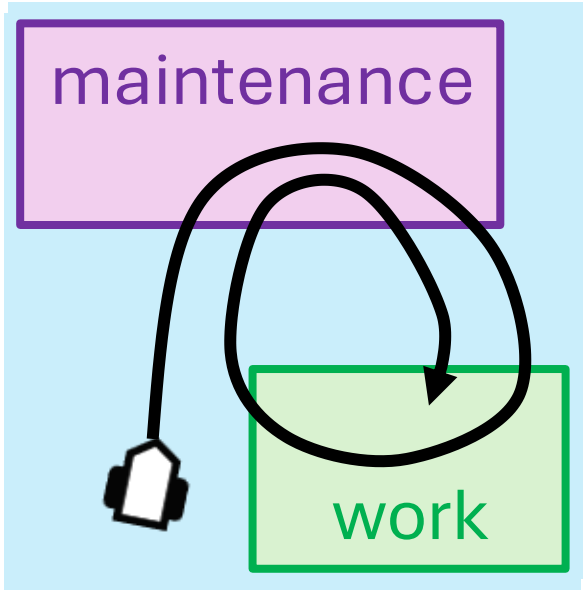
Linear Temporal Logic (LTL)

$$\varphi ::= \underset{\text{(true)}}{\top} \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi U \varphi$$



- $p \in AP$ is an atomic proprosition
  - E.g., "colliding with obstacle", "reaching target region"

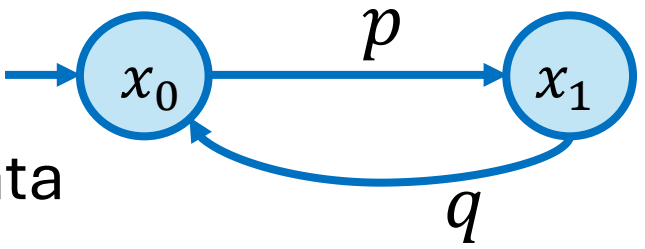
- $\neg obstacle\ U\ target$
- $\Diamond target \equiv \top\ U\ target$
- $\Box\neg obstacle \equiv \neg\Diamond obstacle$
- $\Box\Diamond maintenance\_station \wedge \Box\Diamond work\_station$



LTL as Verification/Control Specifications

Given

- Systems modeled by deterministic finite-state automata

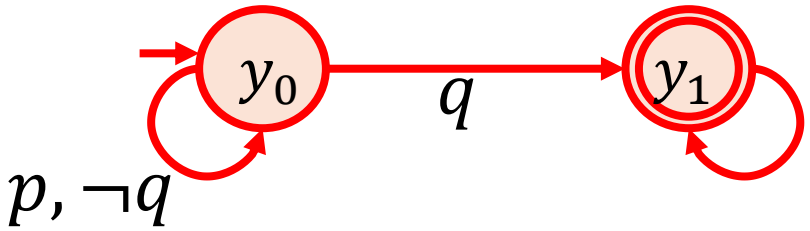


- LTL specification

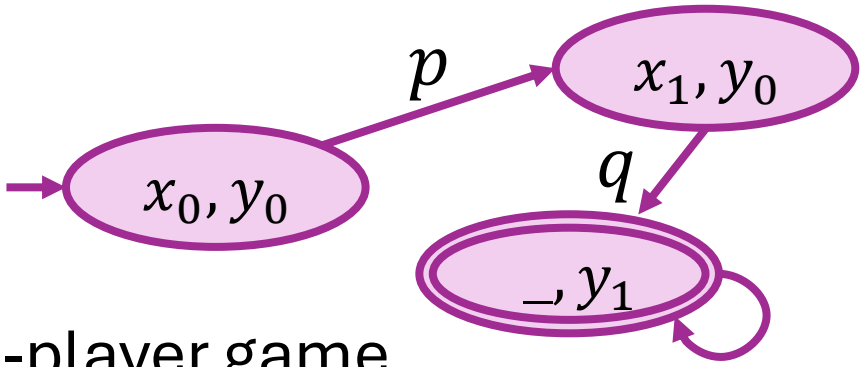
$$pUq$$

(p holds until q holds)

Büchi automaton (Vardi and Wolper, 1986)



System verification problem: by taking product



Controller synthesis problem: formulate as a two-player game

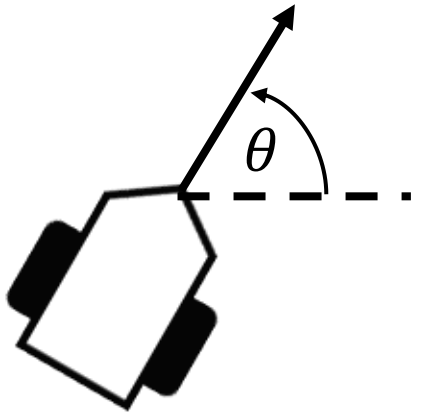
Continuous-state Continuous-time Systems

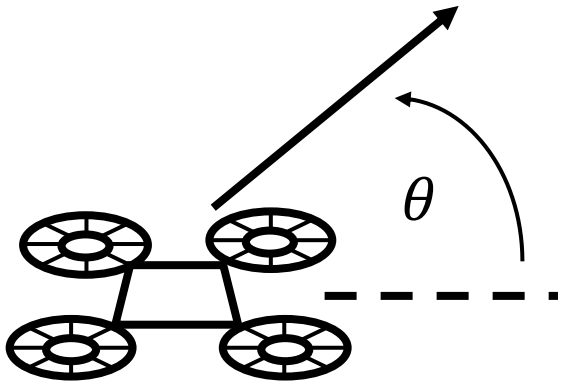
e.g.,

$$\dot{x}(t) = v\big(1 + \lambda(t)\big)\cos(\theta(t))$$

$$\dot{y}(t) = v\big(1 + \lambda(t)\big)\sin(\theta(t))$$

$\dot{\theta}(t)$ is the control input





$$\dot{x}(t) \in [v(t)\cos(\theta(t)) - \varepsilon, v(t)\cos(\theta(t)) + \varepsilon]$$

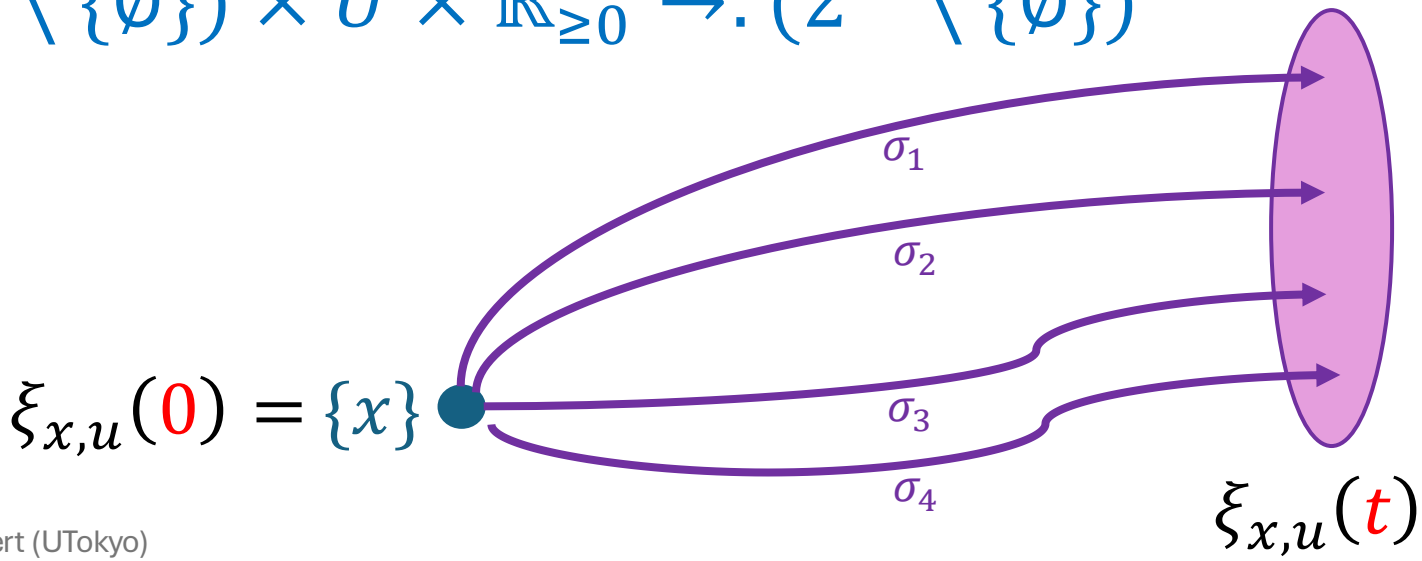
$$\dot{y}(t) \in [v(t)\sin(\theta(t)) - \varepsilon, v(t)\sin(\theta(t)) + \varepsilon]$$

$\theta(t)$ is the control input

Continuous-time Non-deterministic Systems

$$\Sigma = (X, x_{in}, U, \xi)$$

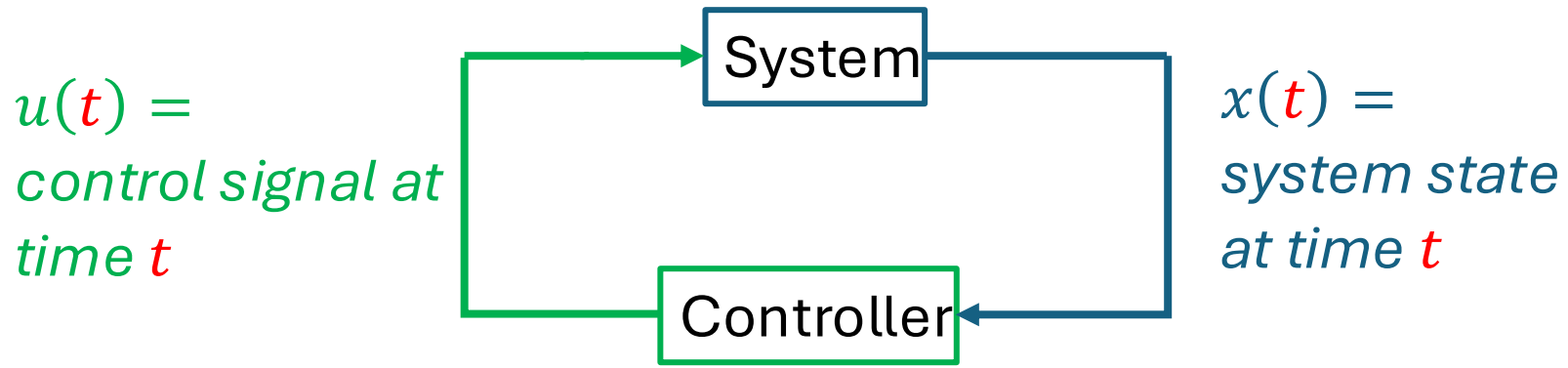
- X: Set of states, $x_{in} \in X$: the initial state
- U: Set of control input
- $\xi: (2^X \setminus \{\emptyset\}) \times U \times \mathbb{R}_{\geq 0} \rightarrow: (2^X \setminus \{\emptyset\})$



Question:

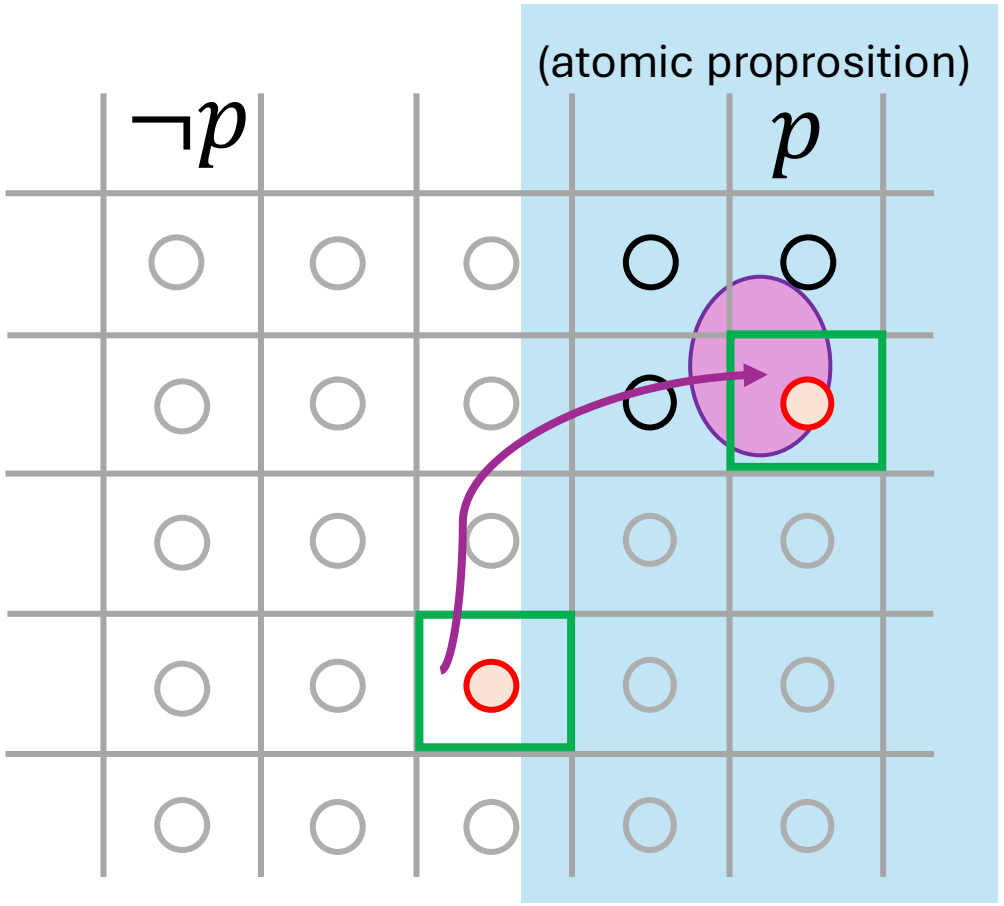
How to verify/control **continuous-time** system for LTL specs?

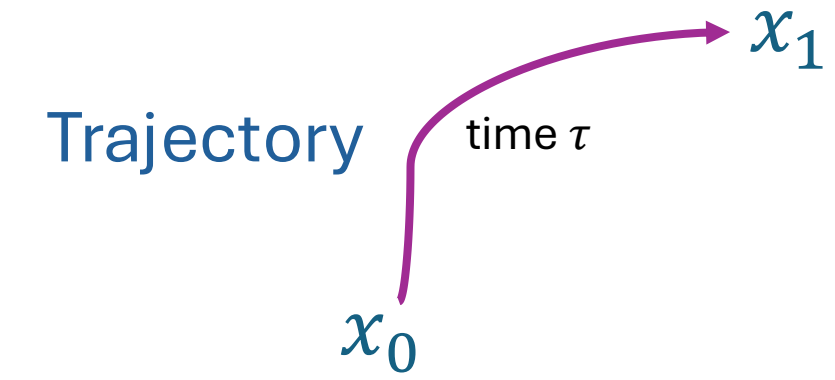
Our focus: Periodic-time Control Loop



$t \in \{0, \tau, 2\tau, \ldots\}, \qquad \tau \in \mathbb{R}_{\geq 0}$

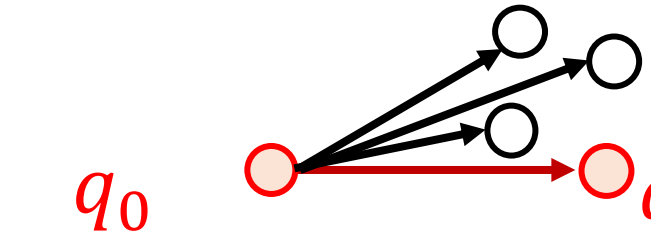
Abstraction: Continuous → Discrete System





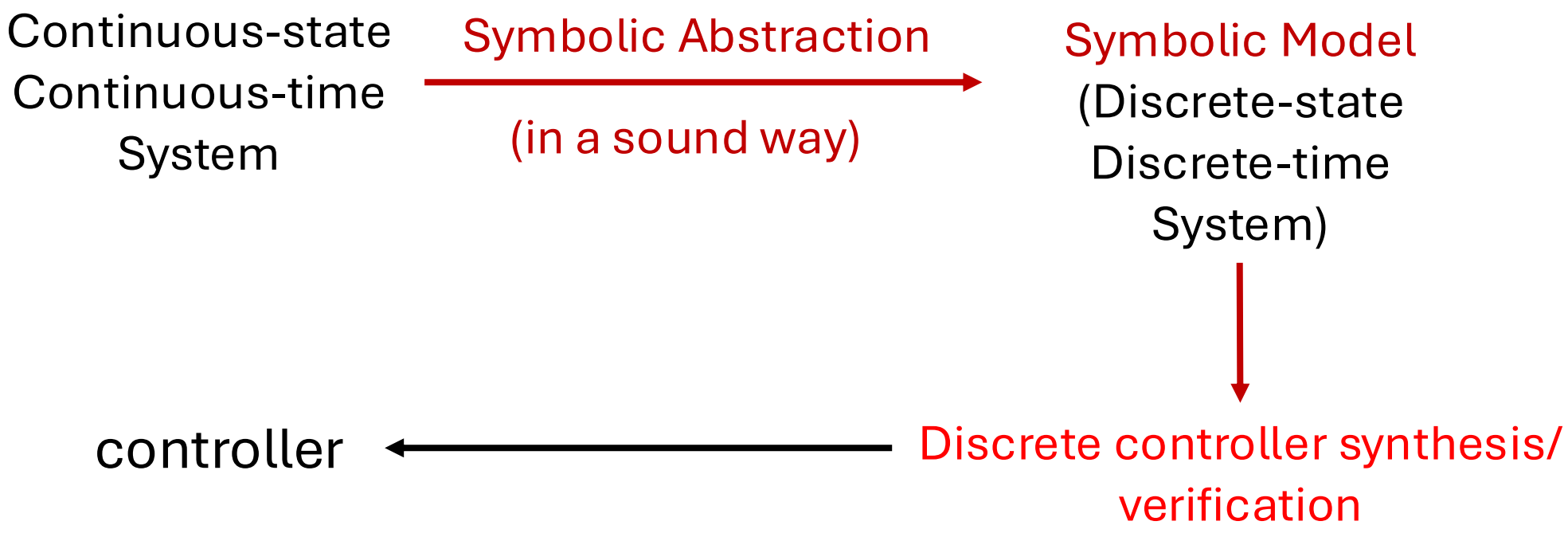
is represented by

discrete transitions



$= abs(x_0)$ $= abs(x_1)$

Abstraction-based Verification & Control



Verification/Control problem is reduced to a problem in a discrete-state space

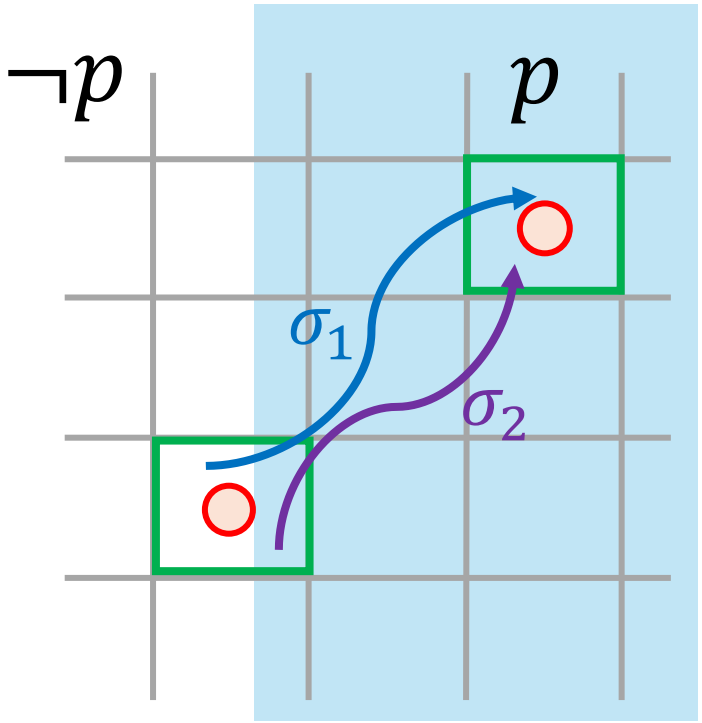
Symbolic Control

- Information loss from abstraction
  - Nondeterminism from the abstraction
  - **Information loss on the atomic propositions from time discretization**

Therefore,
symbolic control often gives sound but not complete solutions

- Previous work consider
  - subclasses of LTL specifications
  - restricted classes of (nonlinear) system
  - discrete-time systems and discrete-time semantics LTL

Information loss on APs from Time discretization



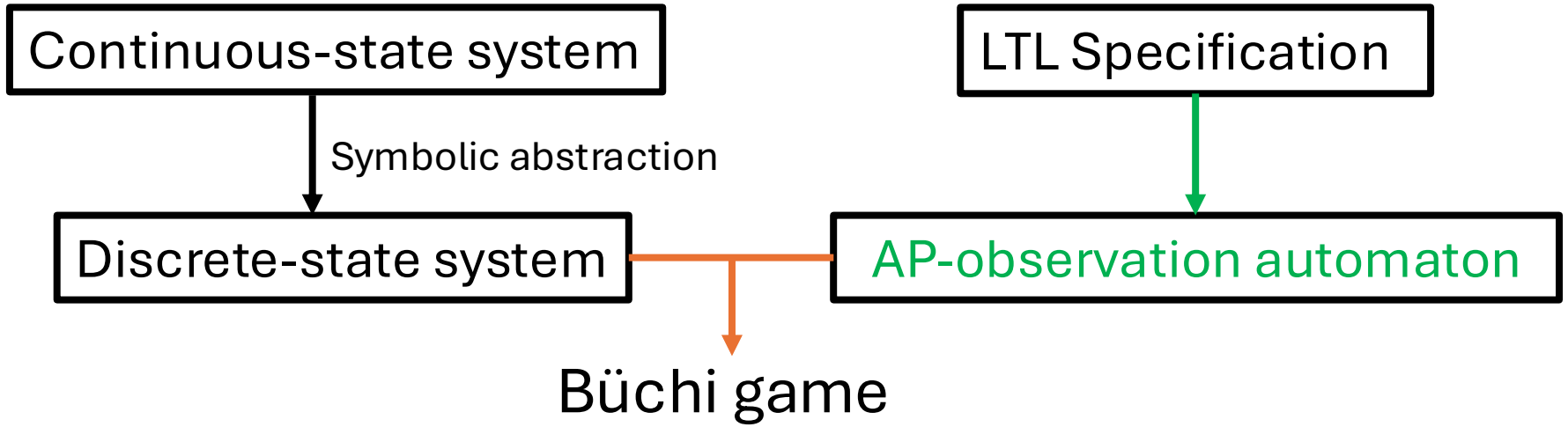
Spec: $\Box p \equiv \neg(TU\neg p)$
(p holds always)

Both trajectories σ_1 and σ_2
are abstracted to
the same transition

However, only σ_2 satisfies $\Box_{[0,\tau]}p$
(p always holds along the trajectory)

Our Contribution

- AP-Observation Automaton: a Büchi automaton tailored specifically for abstraction-based verification & control of continuous-time system



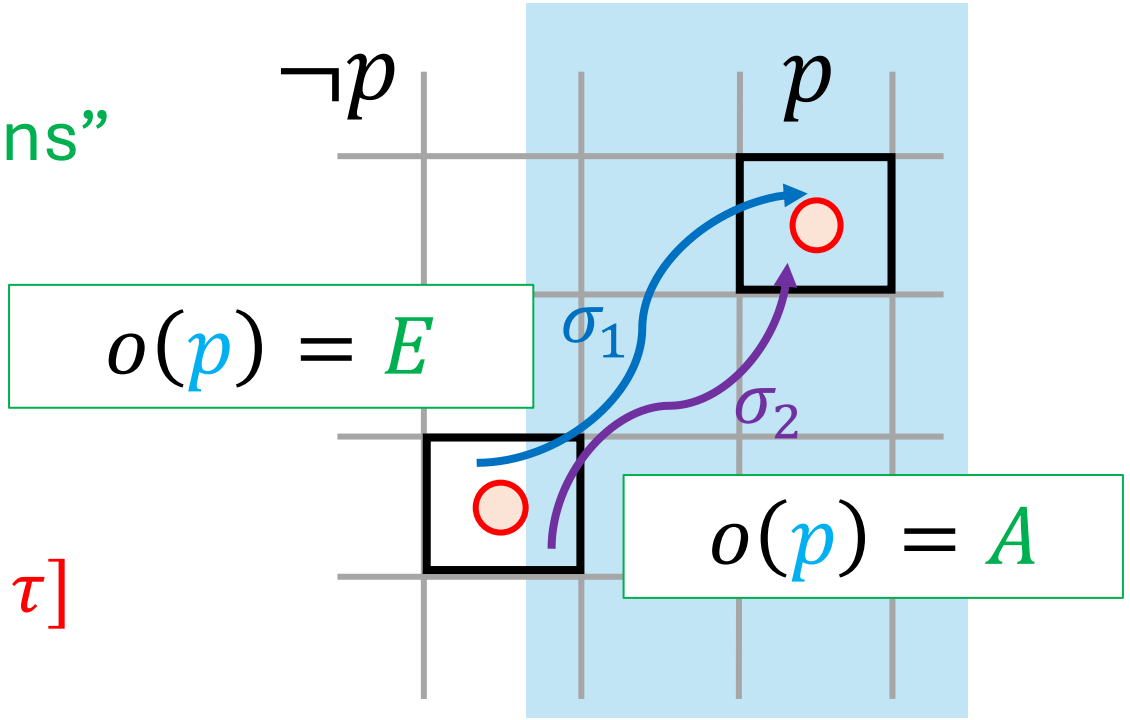
- Two-player game-based verification

Atomic Proposition (AP) Observations

We abstract the satisfaction of

$p \in AP$ along a trajectory $\sigma: [0, \tau] \to X$
into 4 possible "AP-observations"

- A : p holds at All time $t \in [0, \tau]$
- N : p Never hold for all $t \in [0, \tau]$
- Z : $\exists t' \in (0, \tau)$,
  p holds for $[0, t')$, but not for $(t', \tau]$
  (holds at time Zero)
- E : $\exists t' \in (0, \tau)$,
  p holds for $(t', \tau]$, but not for $[0, t')$
  (holds at the End)



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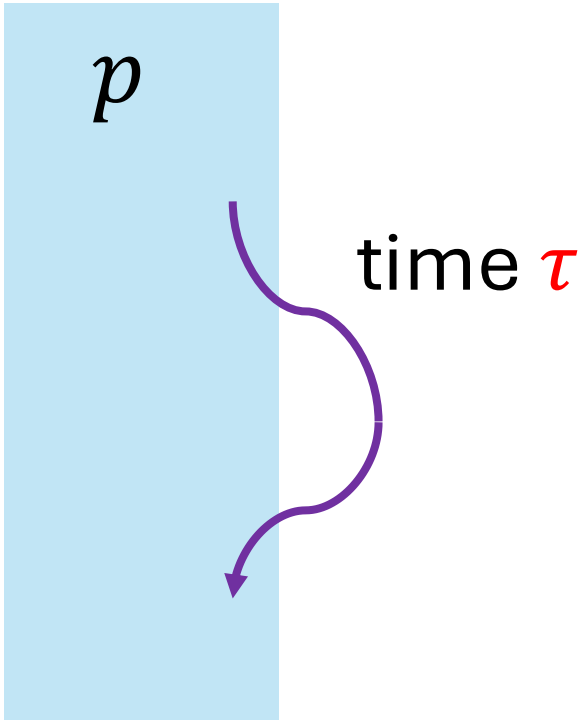

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  p holds for $(t', \tau]$, but not for $[0, t')$
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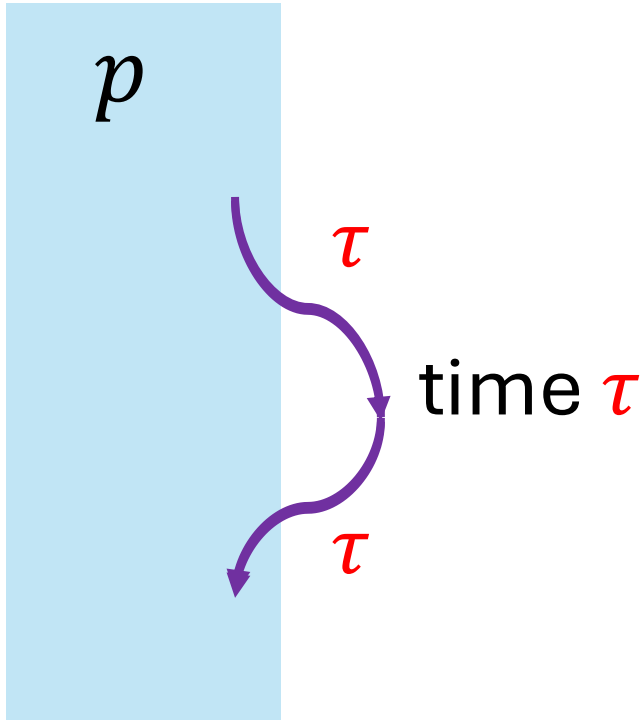
Assumption 1

Given $P: X \to 2^{AP}$ (assigns APs to each state).
$\forall \sigma: [0, \tau] \to X$, $\forall t \in [0, \tau]$, $\forall p \in AP$,

1. If $p \in P(\sigma(0)) \cap P(\sigma(\tau))$,

   then, $p \in P(\sigma(t))$, $\forall t' \in [0, \tau]$

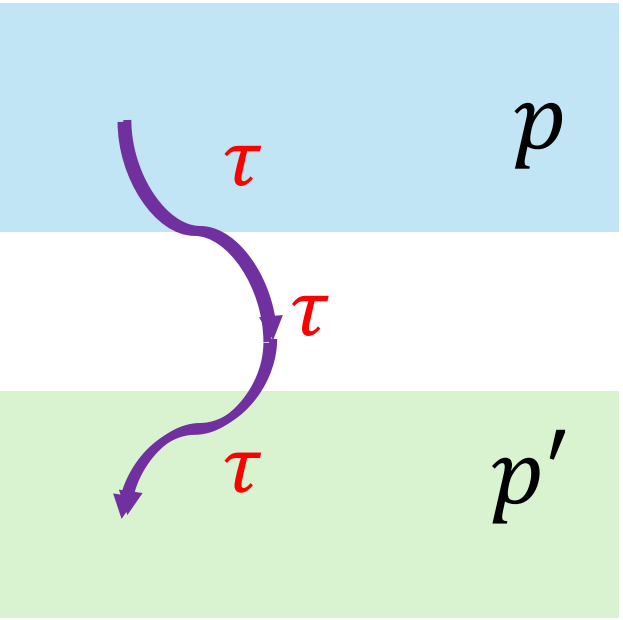


If p holds at the beginning and the end, p holds at all times
(σ cannot cross the border of each AP region twice within time τ)

Assumption 2

Given $P: X \to 2^{AP}$ (assignes APs to each state).
$\forall \sigma: [0, \tau] \to X$, $\forall t \in [0, \tau]$, $\forall p, p' \in AP$,

2. If $p \in P(\sigma(0)) \setminus P(\sigma(\tau))$
   and $p' \in P(\sigma(0)) \setminus P(\sigma(\tau))$,
   then, $p = p'$



(σ cross at most one AP region boundary within time τ)

A constraint for "small enough" τ is **given in Lemma 1**

Atomic Proposition (AP) Observations

We abstract $p \in AP$ along $\sigma: [0, \tau] \to X$ into

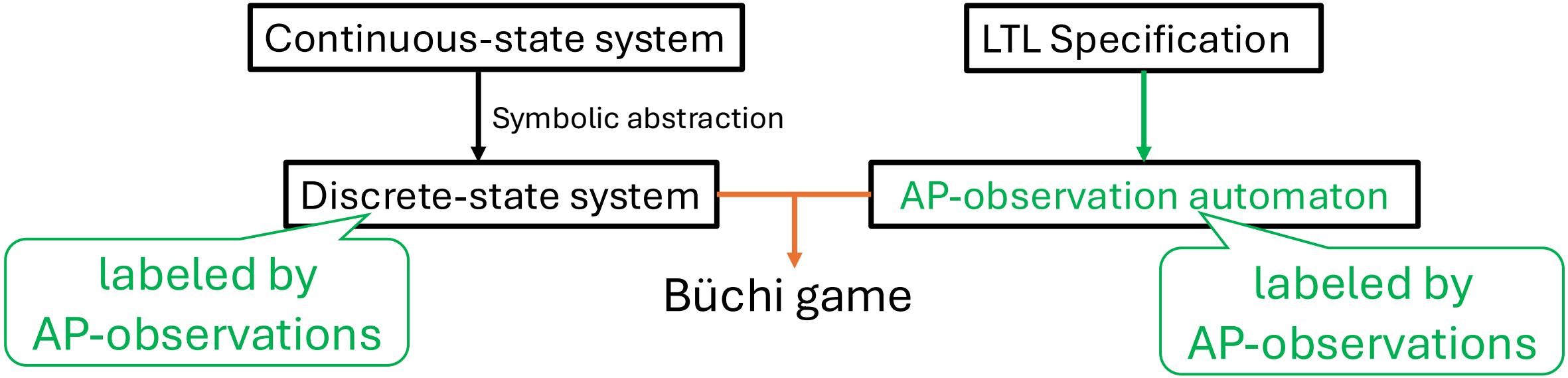
- A : p holds at All time $t \in [0, \tau]$
- N : p Never hold for all $t \in [0, \tau]$
- Z : $\exists t' \in (0, \tau), p$ holds for $[0, t']$, b ero)
- E : $\exists t' \in (0, \tau), p$ holds for $(t', \tau]$, b nd)

Labels of the discrete-state system transitions

By Assumption 1,
for all $\sigma: [0, \tau] \to X$, $o(p) \in \{A, N, Z, E\}$

Our Contribution

- AP-Observation Automaton: a Büchi automaton tailored specifically for abstraction-based verification & control of continuous-time system



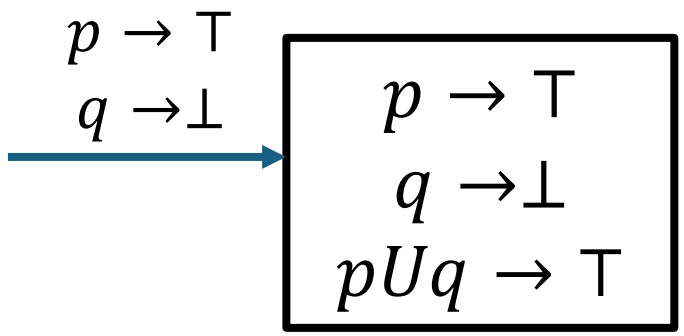
- Two-player game-based verification

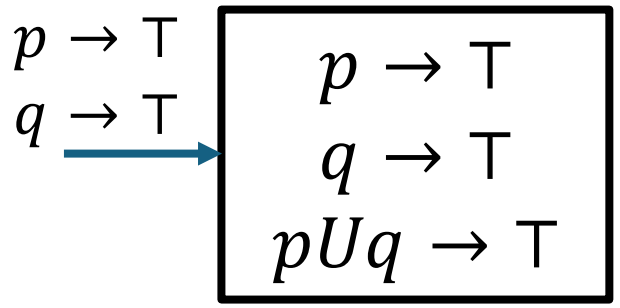
Review: LTL to Buchi automaton (Vardi and Wolper, 1986)

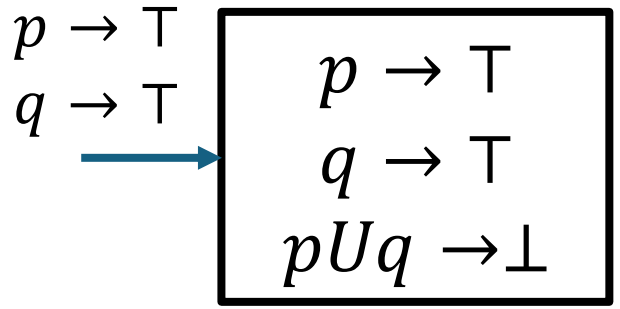
State: sub-formulas → $\{\top, \bot\}$

Transition labels: APs → $\{\top, \bot\}$

e.g., $\varphi = pUq$



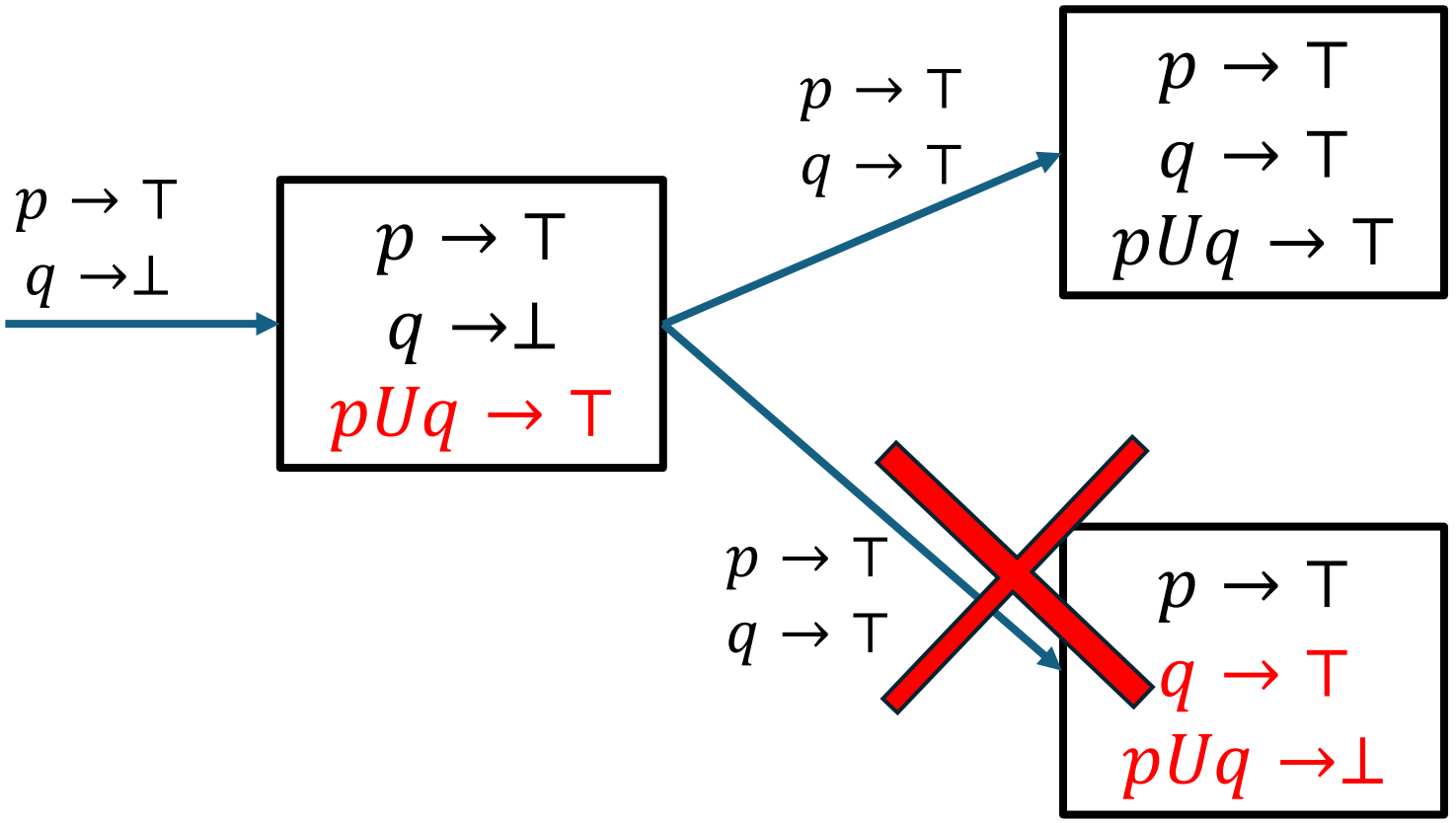


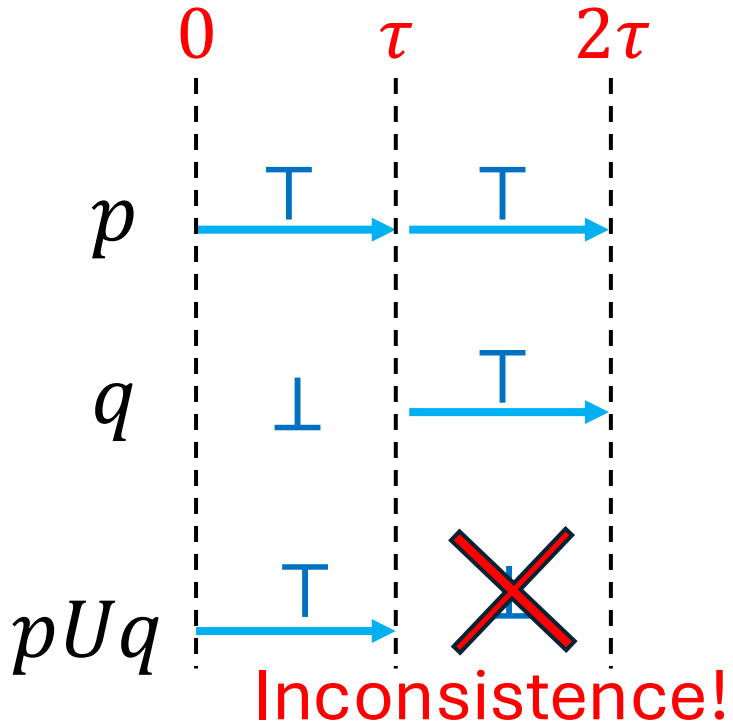


Rule for having transition $M \to M'$:

$pUq \to \top$ *at* M iff ($q \to \top$ *at* M

or ($p \to \top$ *at* M and $pUq \to \top$ *at* M'))

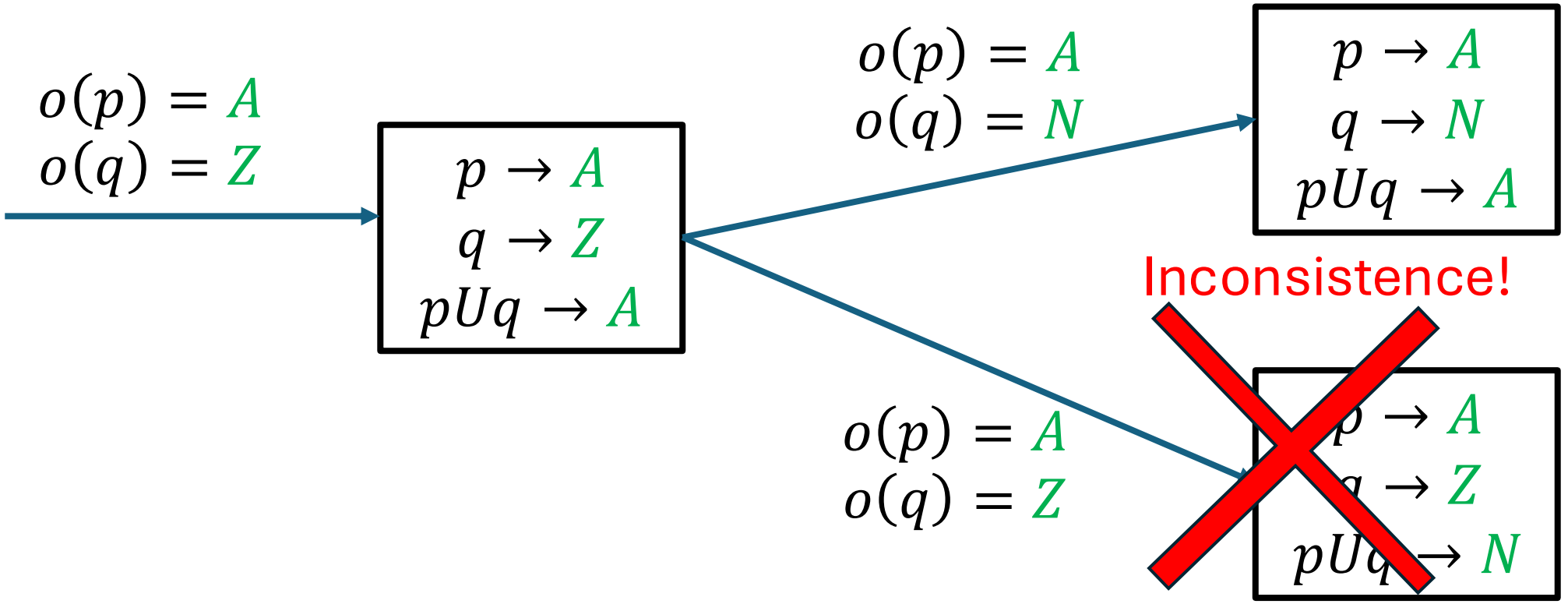




LTL to AP-Observation Automata

State: sub formulas → $\{A, Z, E, N\}$

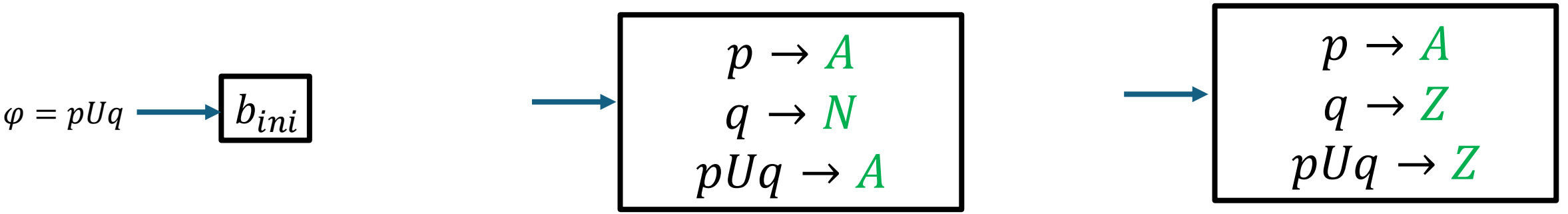
Transition labels: observations of each AP



LTL to AP-Observation Automata

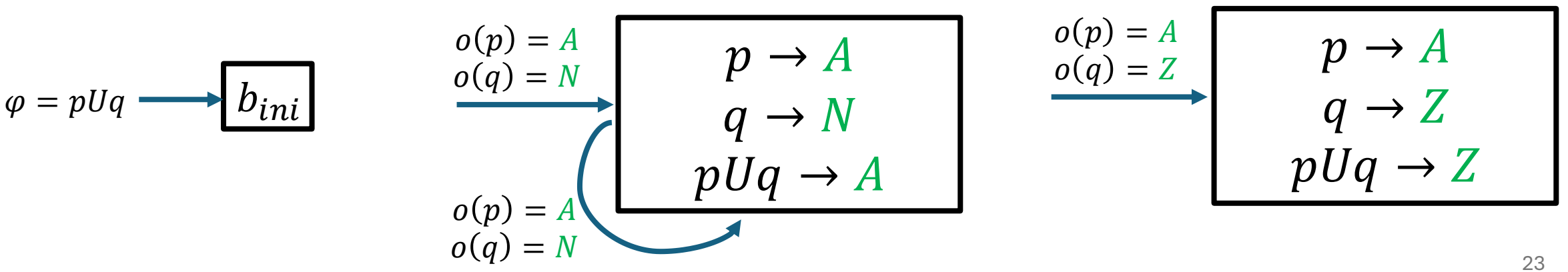
$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

- State set:
  - $cs(\varphi)$ is the set of "consistent subformula valuation"
  - b_{ini} is the initial state



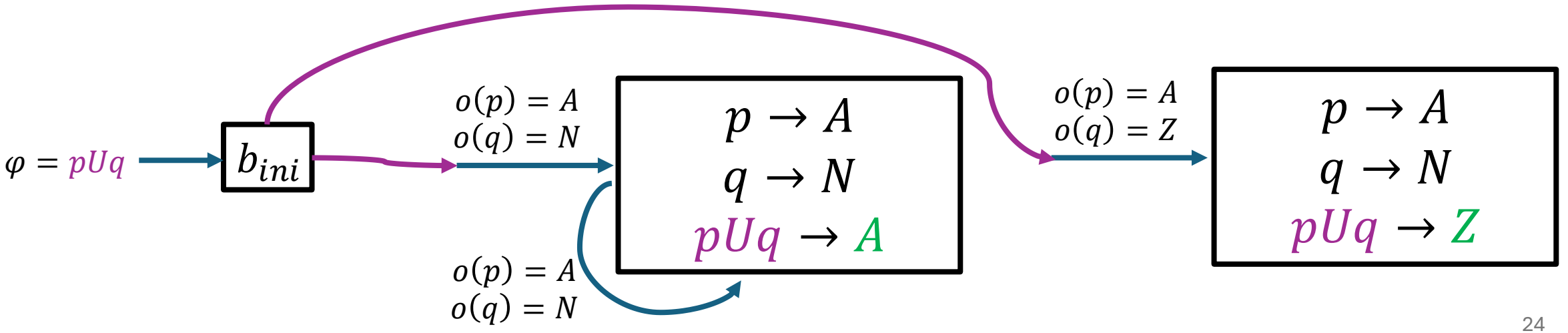
$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

- Transition function δ_φ is labeled by $o: AP \to \{A, Z, E, N\}$
- There exists $M \xrightarrow{o} M'$
  - Forall $a \in AP,\ o(a) = M'(a)$



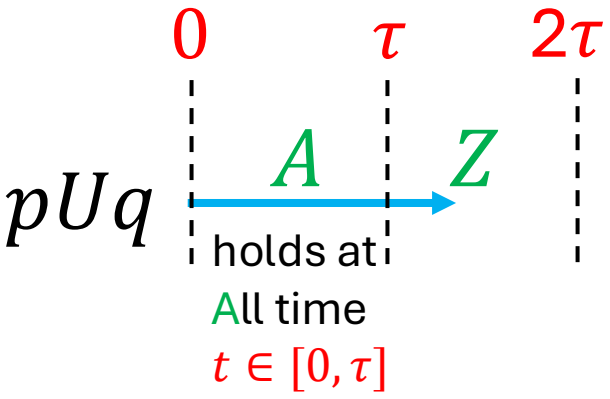
$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

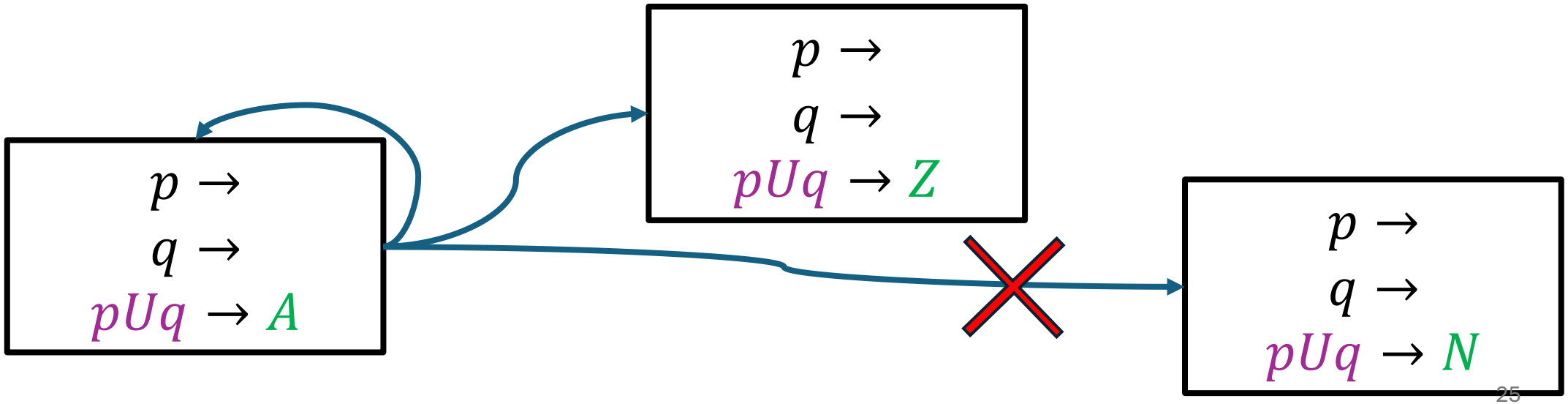
- There exists $M \xrightarrow{o} M'$
  - Forall $a \in AP,\ o(a) = M'(a)$
  - $b_{ini} \xrightarrow{o} M'$ if $M'(\varphi) \in \{A, Z\}$



$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

- There exists $M \xrightarrow{o} M'$
  - Forall $a \in AP,\ o(a) = M'(a)$
  - $b_{ini} \xrightarrow{o} M'$ if $M'(\varphi) \in \{A, Z\}$
- For all sub-formulas ψ, $M(\psi) \in \{A, E\}$ iff $M'(\psi) \in \{A, Z\}$





LTL to AP-Observation Automata

$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

- $cs(\varphi)$ is the set of "**consistent** subformula evaluation"

Which subformulas can hold in the same time interval

ψ_1	ψ_2	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$	$\psi_1 \,\mathrm{U}\, \psi_2$	$\psi_1 \,\mathrm{R}\, \psi_2$
A	A	A	A	A	A
	Z	Z	A	AZ	Z
	E	E	A	A	E
	N	N	A	**AN**	N
Z	A	Z	A	A	AZ
	Z	Z	Z	Z	Z
	E	N	A	A	EN
	N	N	Z	N	N

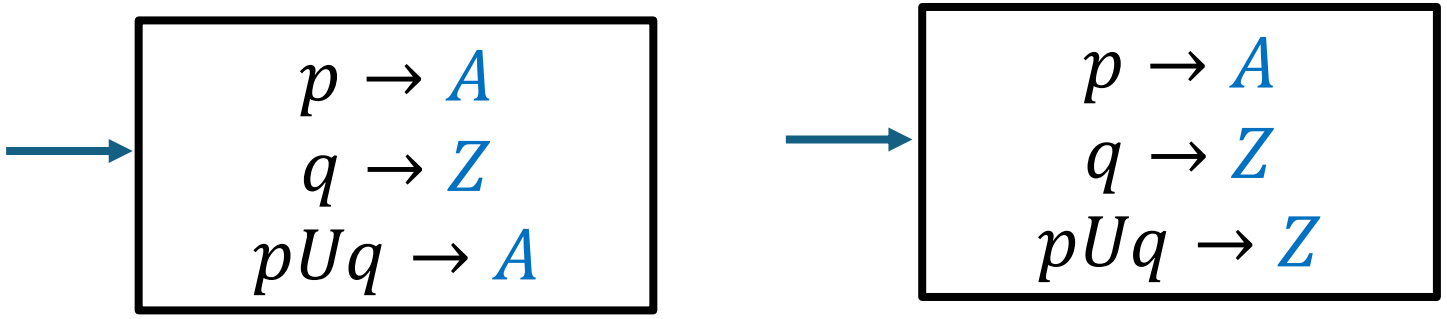
ψ_1	ψ_2	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$	$\psi_1 \,\mathrm{U}\, \psi_2$	$\psi_1 \,\mathrm{R}\, \psi_2$
E	A	E	A	A	A
	Z	N	A	AZ	N
	E	E	E	E	E
	N	N	E	EN	N
N	**A**	N	A	A	**AN**
	Z	N	Z	Z	N
	E	N	E	E	EN
	N	N	N	N	N

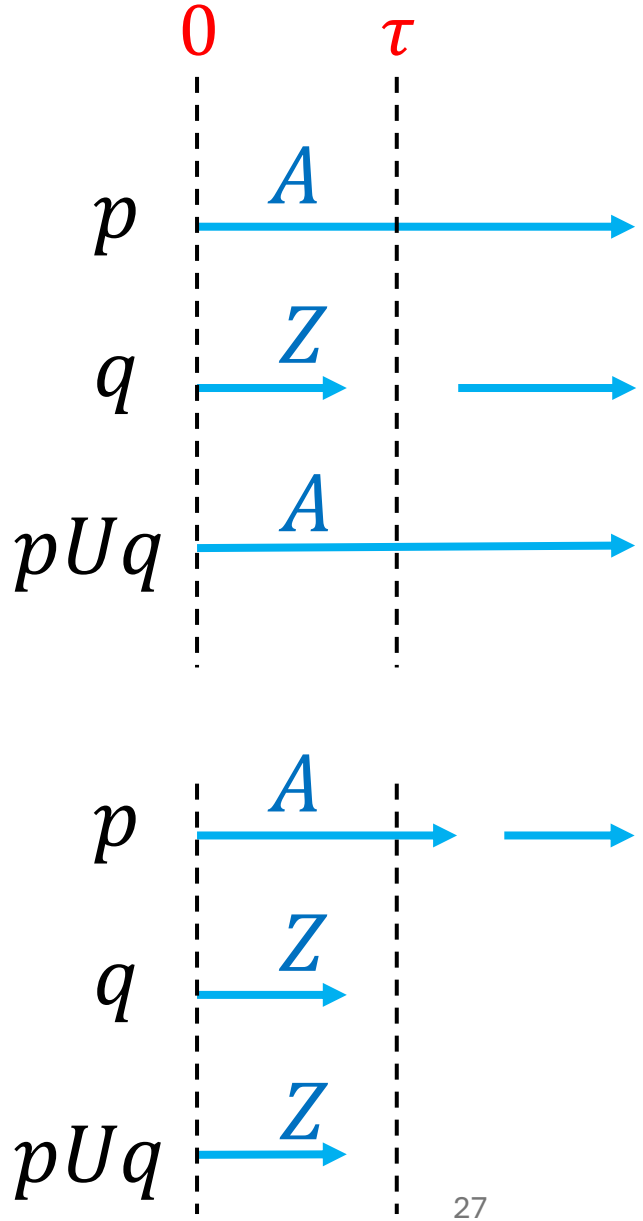
Consistency rules: (ψ_1, ψ_2 are sub-formulas of the specification φ)

LTL to AP-Observation Automata

- $cs(\varphi)$ is the set of "**consistent** subformula evaluation"

p	q			pUq
ψ_1	ψ_2	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$	$\psi_1 \mathrm{U} \psi_2$
A	A	A	A	A
	Z	Z	A	**AZ**
	E	E	A	A
	N	N	A	AN



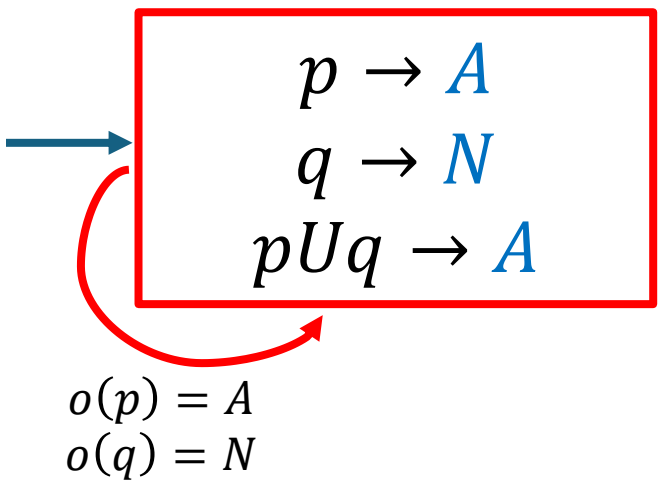


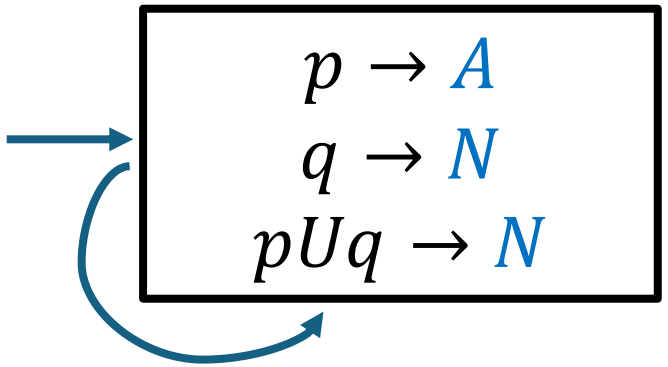
LTL to AP-Observation Automata

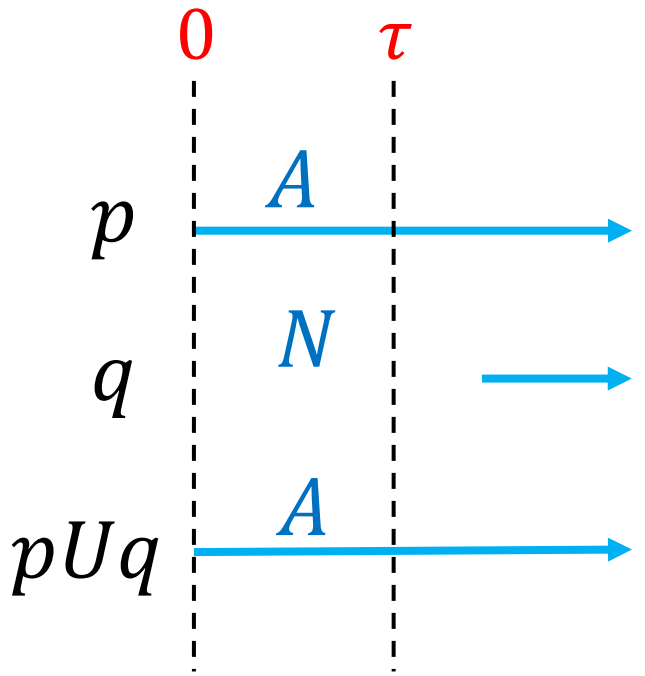
- $cs(\varphi)$ is the set of "**consistent** subformula evaluation"

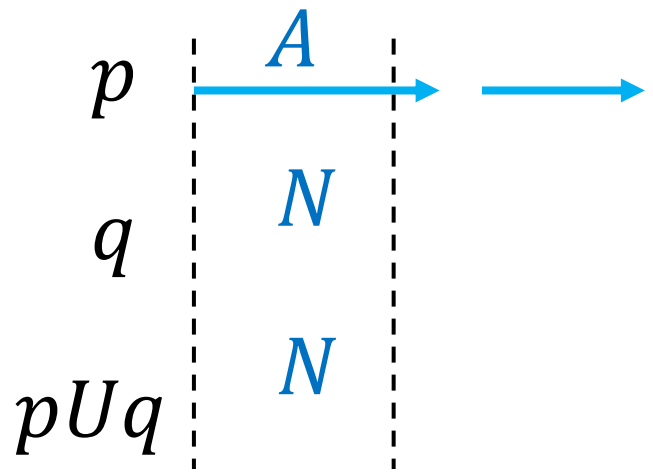
p q pUq

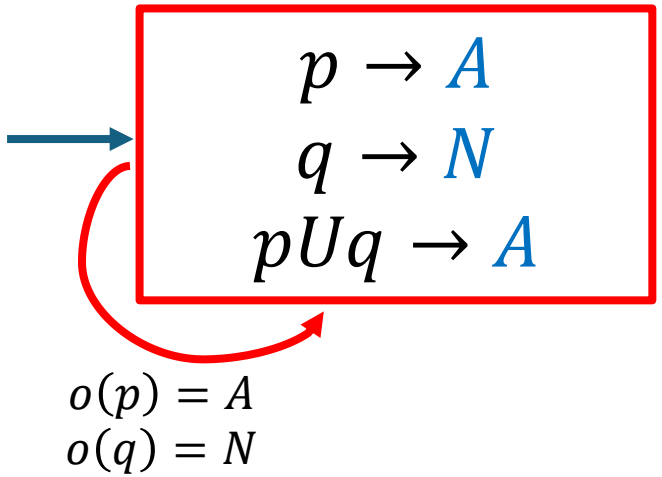
ψ_1	ψ_2	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$	$\psi_1 \mathrm{U} \psi_2$
A	A	A	A	A
	Z	Z	A	AZ
	E	E	A	A
	N	N	A	**AN**





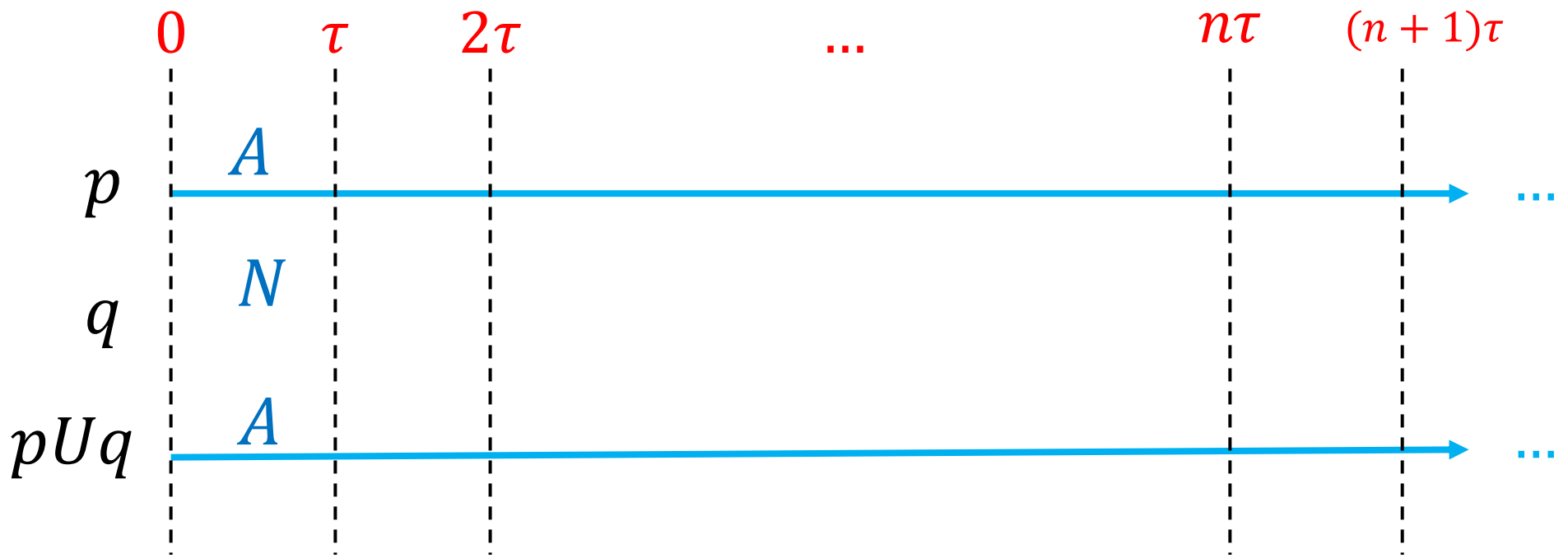






$v \in cs(\varphi)$ is a consistent valuation

$$F_{pUq} = \{v \mid v(q) \neq N \text{ and } v(pUq) \neq A\}$$

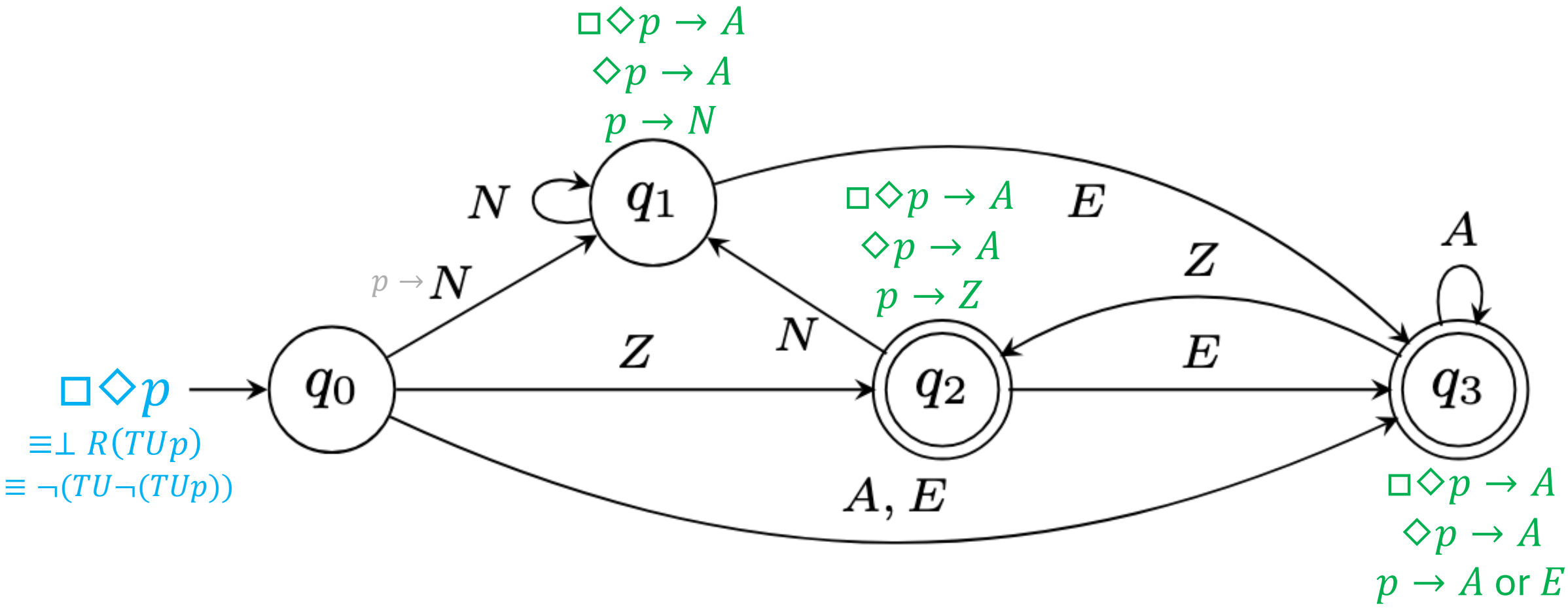


LTL to AP-Observation Automata

$$B_\varphi = (cs(\varphi) \cup \{b_{ini}\}, \delta_\varphi, F_\varphi)$$

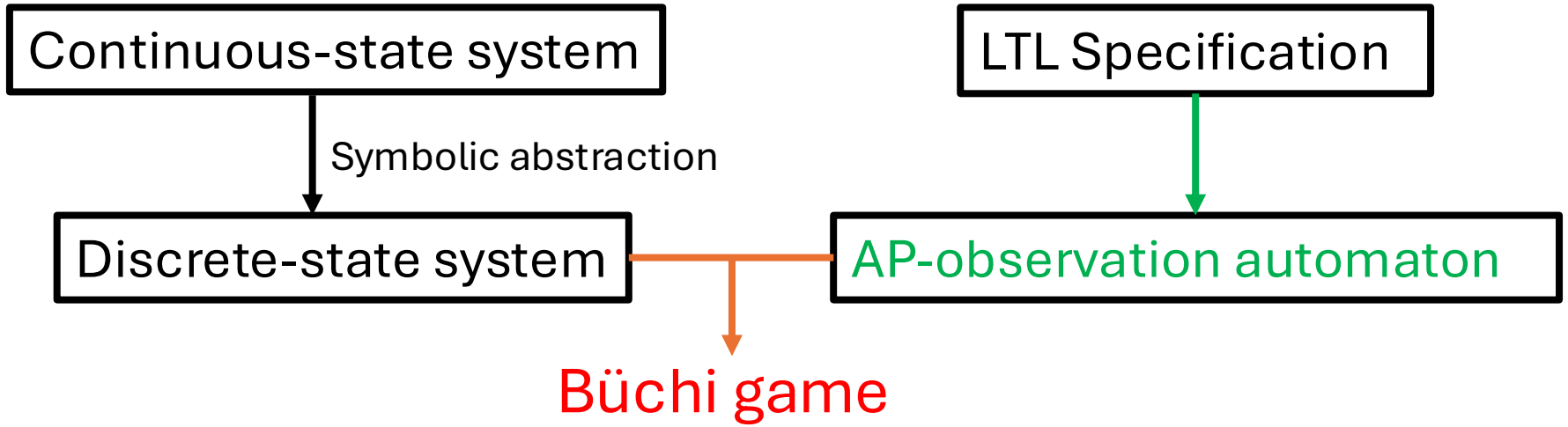
- State set:
  - $cs(\varphi)$ is the set of "consistent subformula evaluation"
  - b_{ini} is the initial state
- Transition function δ_φ is labeled by $o: AP \to \{A, Z, E, N\}$
  There exists $M \xrightarrow{o} M'$
  - Forall $a \in AP,\ o(a) = M'(a)$
  - For all sub-formula ψ, $M(\psi) \in A, E$ iff $M'(\psi) \in A, E$
  - $b_{ini} \xrightarrow{o} M'$ if $M'(\varphi) \in \{A, Z\}$
- Generalized Büchi accepting condition $\{F_{\phi_1}, F_{\phi_2}, \dots, F_{\phi_n}\}$,
  ϕ_i is an $\psi_1 U \psi_2$ or $\psi_1 R \psi_2$ subformula

LTL to AP-Observation Automata

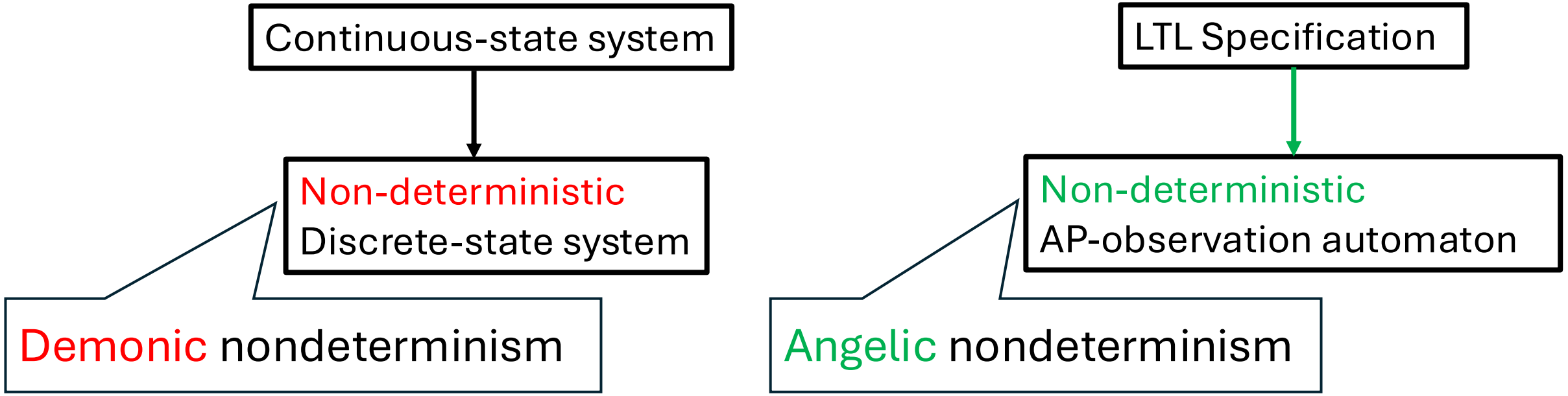


Our Contribution

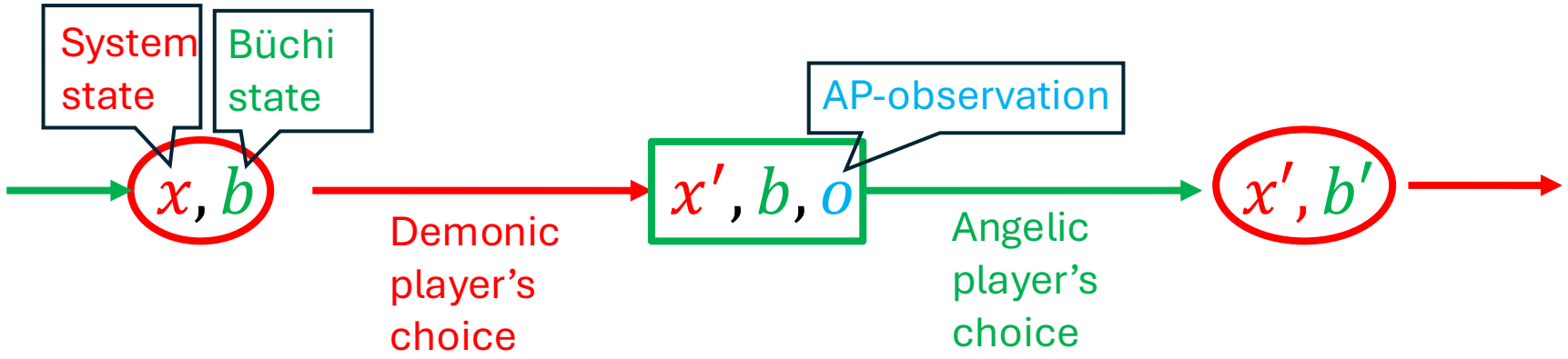
- AP-Observation Automaton: a Büchi automaton tailored specifically for abstraction-based verification & control of continuous-time system



- Two-player game-based verification

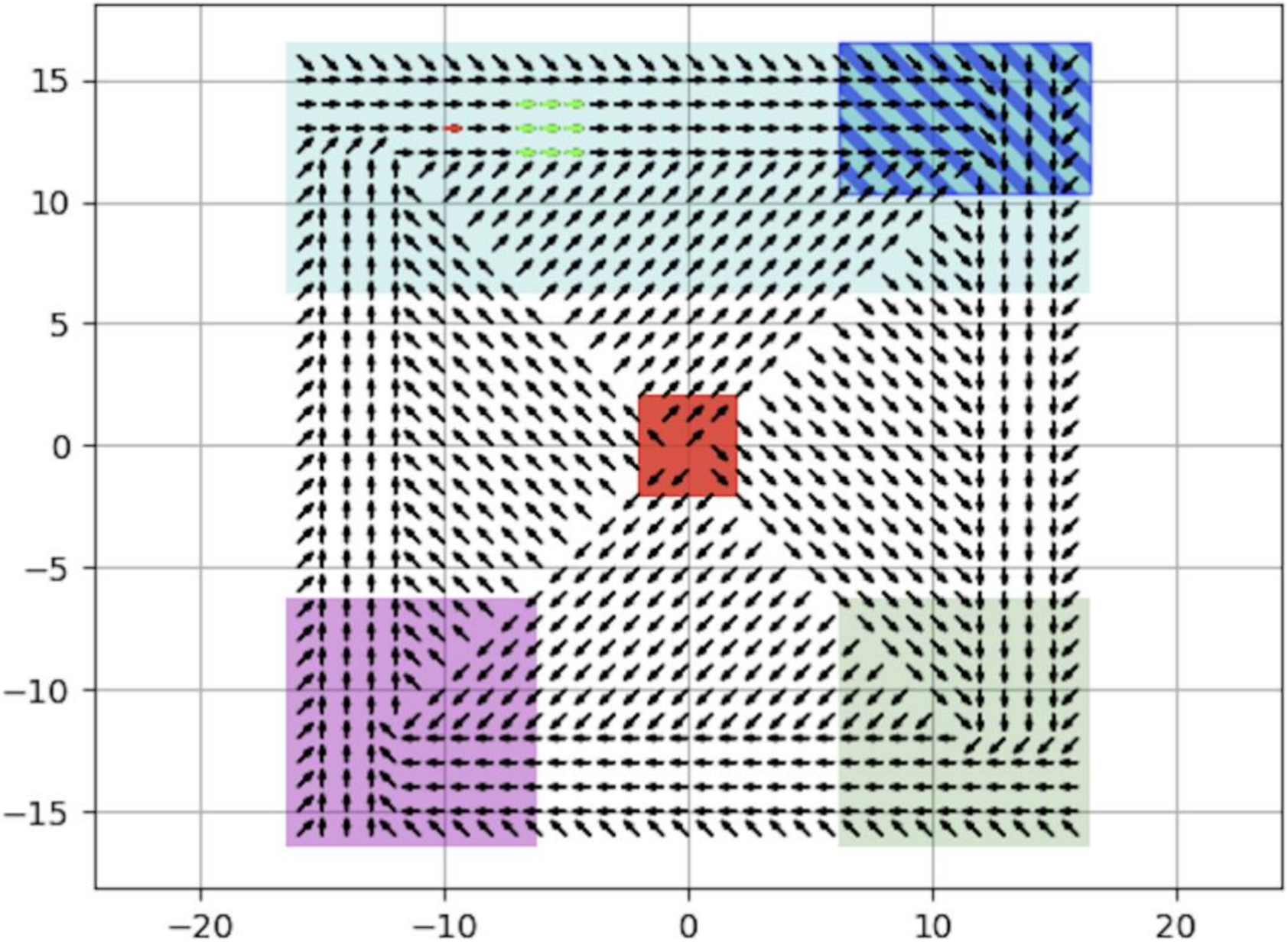


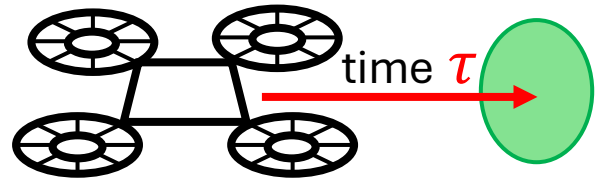
Two-player Büchi Game: Demonic VS Angelic nondeterminism



Theorem:
If the angelic player wins, the system satisfies the specification

Illustrative Example: A surveillance drone

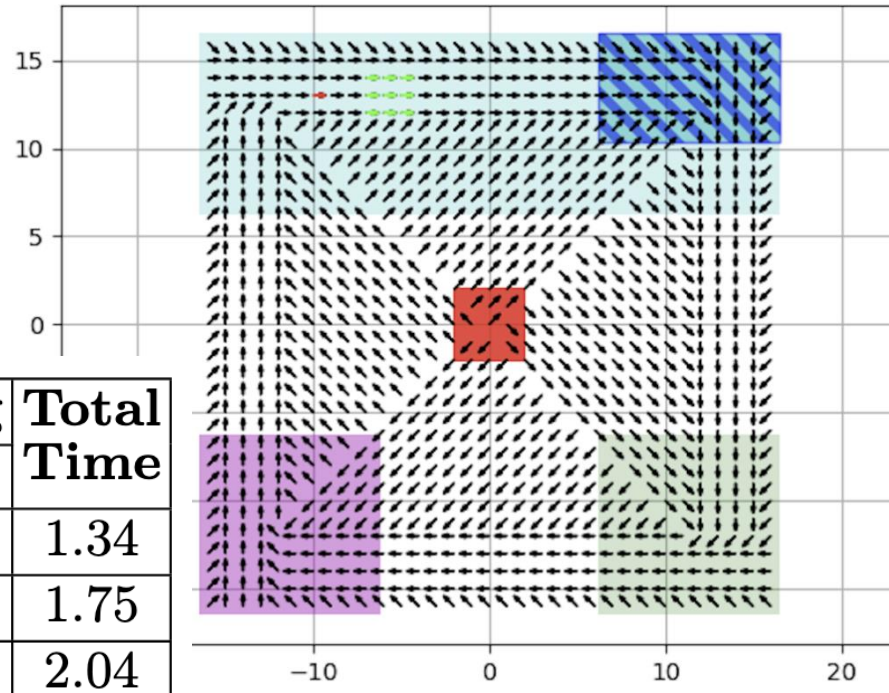




Speed $\in [3.9, 4.1]\ m/s$

Angle for heading direction may vary by up to 0.08 radians

Time interval $\tau = 1s$



(minimized)

Specification φ	$\mathcal{B}_\varphi$		Game construction		Game solving	Total Time
	Size	Time(s)	Size	Time(s)	Time(s)	
$\Box \neg r$	2	0.01	651 + 642	0.23	0.35	1.34
$\Diamond p$	5	0.01	757 + 680	0.52	0.46	1.75
$c \,\mathrm{U}\, b$	7	0.04	830 + 691	0.69	0.55	2.04
$b \,\mathrm{R}\, c$	7	0.04	814 + 685	0.69	0.53	2.03
$\Diamond\Box \neg r$	6	0.02	1,933+1,924	0.60	2.05	3.44
$\Box\Diamond g$	7	0.02	4,640+2,631	1.93	2.48	5.20
$\Diamond(g \wedge \Diamond p)$	33	0.40	4,856+2,687	8.98	3.05	13.19
$\Box \neg r \wedge (\Diamond p \wedge \Diamond c)$	46	51.39	7,723+4,144	29.66	7.54	89.36
$\Box \neg r \wedge \Diamond(g \wedge \Diamond p)$	49	53.86	7,279+4,030	25.49	7.06	87.18

Conclusion

- We consider abstraction-based verification and control of continuous-time continuous-state systems
- We propose
  - AP-Observation Automaton: a Büchi automaton tailored specifically for abstraction-based verification & control of continuous-time system
  - Two-player game-based Verification
- Future work
  - Symbolic control
  - Weaken the assumptions
    consider other type of AP-observations